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ON A MODIFICATION OF THE AVERAGING METHOD FOR SEEKING HIGHER APPROXIMATIONS*

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Systems in the N.N. Bogolyubov standard form as well as systems with rapid phases are considered. It is proposed to seek the solution in the form of an asymptotic series in a small parameter with coefficients representable in the form of the sum of two functions. The first depends on slow time and is found as the solution of a simpler equation in a finite segment. The second is a trigonometric polynomial of the time (or the angular displacements) with coefficients which depend on the slow time (it is found in an explicit manner). It is convenient to use the results in solving certain problems in celestial mechanics.

Utilization of the Bogolyubov-Mitropol'skii-Velosov averaging method /1, 2/ in calculating high approximations of a solution with fixed initial condition can be made complicated because of the awkwardness of appropriate manipulations. A modification is proposed below for the method which is based on ideas utilized in the theory of singularly perturbed equations /3, 4/.

Let R^n be an n-dimensional Euclidean space, and let D be a bounded domain in R^n. We assume that a function $X(t, x)$ with values in R^n, all of whose derivatives with respect to x to the $(N+1)$-th order are continuous, is defined in $[0, \infty) \times D$. Let $X(t, x)$ be a trigonometric polynomial in t.

The Cauchy problem

$$dx/dt = \varepsilon X(t, x),\ x(0) = a \in D,\ t \in [0, T/\varepsilon] \tag{1}$$

is considered, where ε is a small positive parameter. We will seek an approximate solution of this problem in the form

$$x_* = x_0 + \varepsilon x_1 + \ldots + \varepsilon^N x_N$$
$$x_i = u_i(\xi) + v_i(\xi, t),\ i = 0, 1, \ldots, N,\ \xi = \varepsilon t \tag{2}$$

Here v_i are trigonometric polynomials in t. Formally substituting (2) into (1), we have

$$\left[\varepsilon \frac{du_0}{d\xi} + \varepsilon \frac{\partial v_0}{\partial \xi} + \frac{\partial v_0}{\partial t}\right] + \varepsilon\left[\varepsilon \frac{du_1}{d\xi} + \varepsilon \frac{\partial v_1}{\partial \xi} + \frac{\partial v_1}{\partial t}\right] + \ldots = \varepsilon X(t, x_0 + \varepsilon x_1 + \ldots) \tag{3}$$

We shall try to satisfy this equation for all $\xi \in [0, T]$ and $t \in [0, \infty)$. We set $v_0 \equiv 0$. We shall later denote the mean value of the function X with respect to t by $\bar{X}$. Then $X = \bar{X} + X'$. Evidently X' has a zero mean in t. Furthermore

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$$X(t, x_0+\varepsilon x_1+\ldots)=\bar{X}(u_0)+X'(t, u_0)+\varepsilon\left\{\frac{\partial \bar{X}}{\partial x}(u_1+v_1)+\frac{\partial X'}{\partial x}(u_1+v_1)\right\}+\ldots \tag{4}$$

We equate the coefficients of ε in the expansion in (3). Setting

$$du_0/d\xi=\bar{X}(u_0),\ u(0)=\alpha \tag{5}$$

we find $u_0(\xi)\in D$ in a certain segment $[0, T]$ and we obtain the equation $\partial v_1/\partial t = X'[t, u_0(\xi)]$ for v_1. Since X' has a zero mean in t for fixed $\xi\in[0, T]$, then by considering ξ as a parameter, we set $v_1|_{t=0}=0$ and find

$$v_1(\xi, t)=\int_0^t X'[s, u_0(s)]\,ds \tag{6}$$

where $v_1(\xi, t)$ is a trigonometric polynomial in t.

We now equate the coefficients of ε^2 for the expansion in (3). We have

$$\frac{du_1}{d\xi}+\frac{\partial v_2}{\partial t}=\frac{\partial \bar{X}(u_0)}{\partial x}u_1+F_1$$

$$F_1=-\frac{\partial v_1}{\partial \xi}+\left[\frac{\partial \bar{X}(u_0)}{\partial x}v_1+\frac{\partial X'(t, u_0)}{\partial x}u_1+\frac{\partial X'(t, u_0)}{\partial x}v_1\right]$$

The mean value of the function $\partial X'(t, u_0)/\partial x$ with respect to t is evidently zero. Hence, $\bar{F}_1$, the mean value of the function F_1 with respect to t, is defined by the functions u_0 and v_1 already known.

We set

$$\frac{du_1}{d\xi}=\frac{\partial \bar{X}(u_0)}{\partial x}u_1+\bar{F}_1(\xi),\quad u_1(0)=0,\quad 0\leqslant\xi\leqslant T \tag{7}$$

$$\frac{\partial v_2}{\partial t}=F_1-\bar{F}_1 \tag{8}$$

Here u_1 is determined single-valuedly from (7). Consequently, the quantity F_1 is also determined. We can now set

$$v_2(\xi, t)=\int_0^t\{F_1(s, \xi)-\bar{F}_1(\xi)\}\,ds$$

etc. Let all the $x_i=u_i+v_i,\ i=1, 2, \ldots, N$ be defined. Evidently

$$dx_*/dt=\varepsilon X(t, x_*)+f,\ 0\leqslant t\leqslant T/\varepsilon$$

$$(\|f\|\leqslant C\varepsilon^{N+1},\ C>0,\ 0\leqslant t\leqslant T/\varepsilon)$$

To estimate the closeness of the solution x of problem (1) and x_* we should set $w=x-x_*$. We then have

$$dw/dt=\varepsilon\{X(t, x_*+w)-X(t, x_*)-\varepsilon^{-1}f\},\ w(0)=0$$

Changing over to the slow time $\xi=\varepsilon t$, we obtain the equation

$$w=\int_0^\xi\left\{X\left(\frac{s}{\varepsilon}, x_*+w\right)-X\left(\frac{s}{\varepsilon}, x_*\right)-\varepsilon^{-1}f\right\}ds,\ \xi=\varepsilon t$$

From the principle of compressed mapping we have.

Theorem. Under the above assumptions numbers $C>0$ and $\varepsilon_0>0$ exist such that for all $\varepsilon\in(0, \varepsilon_0]$ a solution exists for the Cauchy problem (1) for $t\in[0, T/\varepsilon]$ and

$$\sup_{t\leqslant T\leqslant T/\varepsilon}\|x(t, \varepsilon)-x_*\|\leqslant C\varepsilon^N \tag{9}$$

Now, let R^m be an m-dimensional Euclidean space and the functions $X(x, y)$ and $Y(x, y)$ be determined in $\bar{D}\times R^m$ with values in R^n and R^m, respectively. Let $\omega: \bar{D}\to R^m$. We shall consider that X and Y are trigonometric polynomials in $y=(y_1, \ldots, y_m)$ of period 2π. Let X, Y and ω be $(N+1)$ times continuously differentiable with respect to x and y.

The Cauchy problem

$$dx/dt=\varepsilon X(x, y),\ dy/dt=\omega(x)+\varepsilon Y(x, y) \tag{10}$$

$$x(0)=\alpha\in D,\ y(0)=\beta\in R^m,\ 0\leqslant t\leqslant T/\varepsilon$$

is considered with a small positive parameter ε.

We will seek the approximate solution of this problem in the form

$$x_*=x_0+\varepsilon x_1+\ldots+\varepsilon^N x_N,\ y_*=\beta+\psi+\varepsilon y_1+\ldots+\varepsilon^N y_N$$

$$x_i=u_i(\xi)+v_i(\xi, \psi),\ v_0\equiv 0,\ y_i=y_i(\xi, \psi),\ i=1, 2, \ldots, N,\ \xi=\varepsilon t$$

$$\psi=\int_0^t\sum_0^N \varepsilon^s\omega_s(\varepsilon, \tau)\,d\tau$$

Therefore, $\psi' = \Omega = \Sigma \varepsilon^s \omega_s(\xi)$. As before, we denote the mean values of X and Y with respect to the variable y by $\bar{X}, \bar{Y}$. Let $X' = X - \bar{X}, Y' = Y - \bar{Y}$.

Evidently

$$\frac{dx}{dt} = \varepsilon \frac{du_0}{d\xi} + \varepsilon \left[\varepsilon \frac{du_1}{d\xi} + \varepsilon \frac{\partial v_1}{\partial \xi} + \frac{\partial v_1}{\partial \psi} \Omega \right] + \ldots \tag{11}$$

$$\frac{dy}{dt} = \Omega + \varepsilon \left[\varepsilon \frac{\partial y_1}{\partial \xi} + \frac{\partial y_1}{\partial \psi} \Omega \right] + \ldots \tag{12}$$

Furthermore

$$X(x_0 + \varepsilon x_1 + \ldots, \beta + \psi + \varepsilon y_1 + \ldots) = \bar{X}(u_0) + X'(u_0, \beta + \psi) + \varepsilon \left\{ \frac{\partial \bar{X}}{\partial x}(u_1 + v_1) + \frac{\partial \bar{X}}{\partial y} y_1 + \frac{\partial X'}{\partial x}(u_1 + v_1) + \frac{\partial X'}{\partial y} y_1 \right\} + \ldots \tag{13}$$

$$Y(x_0 + \varepsilon x_1 + \ldots, \beta + \psi + \varepsilon y_1 + \ldots) = \bar{Y}(u_0) + Y'(u_0, \beta + \psi) + \left\{ \left(\frac{\partial \bar{Y}}{\partial x} + \frac{\partial Y'}{\partial x} \right)(u_1 + v_1) + \left(\frac{\partial \bar{Y}}{\partial y} + \frac{\partial Y'}{\partial y} \right) y_1 \right\} + \ldots \tag{14}$$

As before, we determine the function $u_0(\xi)$ by solving a problem analogous to (5). We set $\omega_0(\xi) = \omega[u_0(\xi)]$. Then we have the following relationship for v_1

$$\frac{\partial v_1}{\partial \psi} \omega_0 = X'(u_0, \beta + \psi) \tag{15}$$

If it is assumed that for any $\xi \in [0, T]$ and any integer vector $k = (k_1, \ldots, k_m) \neq 0$

$$(k, \omega_0(\xi)) \neq 0 \tag{16}$$

then from (15) the Fourier coefficients of the function $v_1(\xi, \psi)$ are determined single-valuedly (if it is considered that the mean value of $v_1(\xi, \psi)$ with respect to ψ is zero).

We henceforth define v_i and y_i in such a manner that their mean value with respect to ψ is zero.

We then have from (10), (12), and (14)

$$\omega_1 + \frac{\partial y_1}{\partial \psi} \omega_0 = \frac{\partial \omega(u_0)}{\partial x}(u_1 + v_1) + \bar{Y}(u_0) + Y'(u_0, \beta + \psi)$$

For y_1 and ω_1 we obtain the relationships

$$\frac{\partial y_1}{\partial \psi} \omega_0 = \frac{\partial \omega(u_0)}{\partial x} v_1 + Y'(u_0, \beta + \psi), \quad \omega_1 = \frac{\partial \omega(u_0)}{\partial x} u_1 + \bar{Y}(u_0) \tag{17}$$

Hence, y_1 is determined single-valuedly, while ω_1 will be refined later. Furthermore, we have from (10), (11) and (14)

$$\frac{du_1}{d\xi} + \frac{\partial v_1}{\partial \xi} + \frac{\partial v_1}{\partial \psi} \omega_1 + \frac{\partial v_2}{\partial \psi} \omega_0 = \left(\frac{\partial \bar{X}}{\partial x} + \frac{\partial X'}{\partial x} \right)(u_1 + v_1) + \left(\frac{\partial \bar{X}}{\partial y} + \frac{\partial X'}{\partial y} \right) y_1 \tag{18}$$

We note that the series of components has a zero mean with respect to ψ. Hence, we obtain equations for u_1 and v_2

$$\frac{du_1}{d\xi} = \frac{\partial \bar{X}(u_0)}{\partial x} u_1 + U_1(\xi), \quad \frac{\partial v_2}{\partial \psi} \omega_0 = V_2(\xi, \psi) \tag{19}$$

in which U_1 is an already known function and the mean value of V_2 with respect to ψ equals zero.

The $u_1(\xi)$ with the initial condition $u_1(0) = -v_1(0, 0)$ is determined single-valuedly from the first equation in (19). Then we find ω_1 from the second equation in (17) and we refine the function $v_2(\xi, \psi)$ from (18). Finally, $V_2(\xi, \psi)$ is determined single-valuedly from the second equation in (19). Subsequent terms are determined by the same scheme.

Having determined N terms of the series, we obtain uniform estimates in $t \in [0, T/\varepsilon]$

$$\| dx_*/dt - \varepsilon X(x_*, y_*) \| \leqslant C\varepsilon^{N+1}$$
$$\| dy_*/dt - \omega(x_*) - \varepsilon Y(x_*, y_*) \| \leqslant C\varepsilon^N$$

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